Starting at the output we obtain

$$I(s) = G_1(s)G_2(s)E(s).$$

But $E(s) = R(s) - H(s)I(s)$, so

$$I(s) = G_1(s)G_2(s)\left[R(s) - H(s)I(s)\right].$$

Solving for $I(s)$ yields the closed-loop transfer function

$$\frac{I(s)}{R(s)} = \frac{G_1(s)G_2(s)}{1 + G_1(s)G_2(s)H(s)} .$$

E2.8 The block diagram is shown in Figure E2.8.

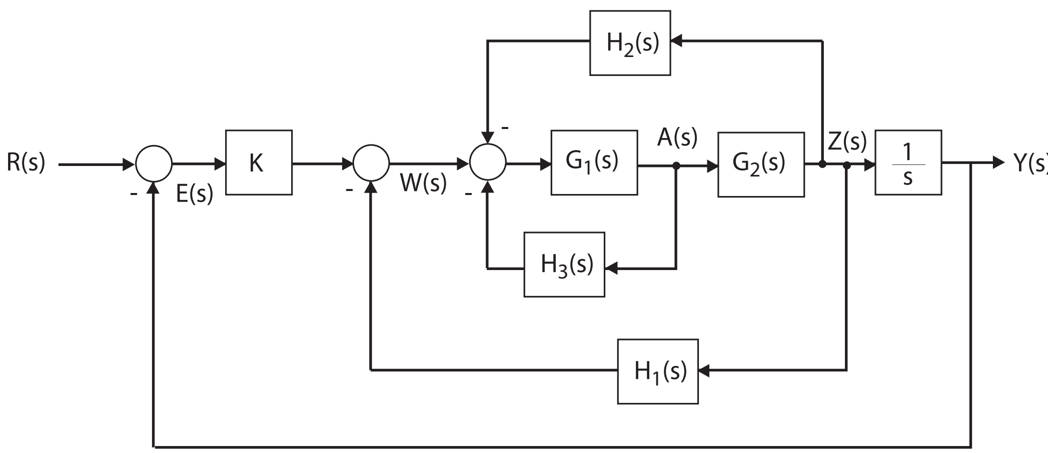

FIGURE E2.8
Block diagram model.

Starting at the output we obtain

$$Y(s) = \frac{1}{s}Z(s) = \frac{1}{s}G_2(s)A(s).$$

But $A(s) = G_1(s)\left[-H_2(s)Z(s) - H_3(s)A(s) + W(s)\right]$ and $Z(s) = sY(s)$, so

$$Y(s) = -G_1(s)G_2(s)H_2(s)Y(s) - G_1(s)H_3(s)Y(s) + \frac{1}{s}G_1(s)G_2(s)W(s).$$

Substituting $W(s) = KE(s) - H_1(s)Z(s)$ into the above equation yields

$$\begin{aligned} Y(s) = &-G_1(s)G_2(s)H_2(s)Y(s) - G_1(s)H_3(s)Y(s) \\ &+ \frac{1}{s}G_1(s)G_2(s)\left[KE(s) - H_1(s)Z(s)\right] \end{aligned}$$

and with $E(s) = R(s) - Y(s)$ and $Z(s) = sY(s)$ this reduces to

$$Y(s) = [-G_1(s)G_2(s)(H_2(s) + H_1(s)) - G_1(s)H_3(s) \\ - \frac{1}{s}G_1(s)G_2(s)K]Y(s) + \frac{1}{s}G_1(s)G_2(s)KR(s).$$

Solving for $Y(s)$ yields the transfer function

$$Y(s) = T(s)R(s),$$

where

$$T(s) = \frac{KG_1(s)G_2(s)/s}{1 + G_1(s)G_2(s)[(H_2(s) + H_1(s)] + G_1(s)H_3(s) + KG_1(s)G_2(s)/s}.$$

E2.9 From Figure E2.9, we observe that

$$F_f(s) = G_2(s)U(s)$$

and

$$F_R(s) = G_3(s)U(s) \ .$$

Then, solving for $U(s)$ yields

$$U(s) = \frac{1}{G_2(s)}F_f(s)$$

and it follows that

$$F_R(s) = \frac{G_3(s)}{G_2(s)}U(s) \ .$$

Again, considering the block diagram in Figure E2.9 we determine

$$F_f(s) = G_1(s)G_2(s)[R(s) - H_2(s)F_f(s) - H_2(s)F_R(s)] \ .$$

But, from the previous result, we substitute for $F_R(s)$ resulting in

$$F_f(s) = G_1(s)G_2(s)R(s) - G_1(s)G_2(s)H_2(s)F_f(s) - G_1(s)H_2(s)G_3(s)F_f(s) \ .$$

Solving for $F_f(s)$ yields

$$F_f(s) = \left[\frac{G_1(s)G_2(s)}{1 + G_1(s)G_2(s)H_2(s) + G_1(s)G_3(s)H_2(s)}\right]R(s) \ .$$

P2.28 The signal flow graph is shown in Figure P2.28.

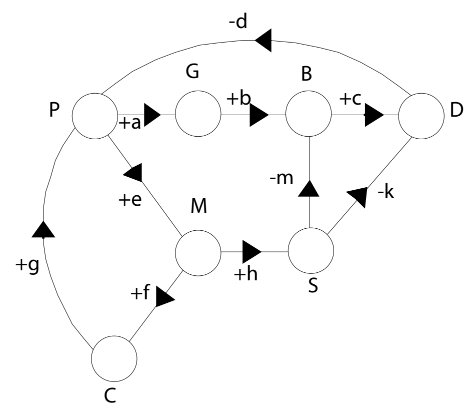

FIGURE P2.28
Signal flow graph.

(a) The PGBDP loop gain is equal to -abcd. This is a **negative** transmission since the population produces garbage which increases bacteria and leads to diseases, thus reducing the population.

(b) The PMCP loop gain is equal to +efg. This is a **positive** transmission since the population leads to modernization which encourages immigration, thus increasing the population.

(c) The PMSDP loop gain is equal to +ehkd. This is a **positive** transmission since the population leads to modernization and an increase in sanitation facilities which reduces diseases, thus reducing the rate of decreasing population.

(d) The PMSBDP loop gain is equal to +ehmcd. This is a **positive** transmission by similar argument as in (3).

P2.29 Assume the motor torque is proportional to the input current

$$T_m = ki \ .$$

Then, the equation of motion of the beam is

$$J\ddot{\phi} = ki \ ,$$

where J is the moment of inertia of the beam and shaft (neglecting the inertia of the ball). We assume that forces acting on the ball are due to gravity and friction. Hence, the motion of the ball is described by

$$m\ddot{x} = mg\phi - b\dot{x}$$

P2.32 From the block diagram we have

$$\begin{aligned} Y_1(s) &= G_2(s)[G_1(s)E_1(s) + G_3(s)E_2(s)] \\ &= G_2(s)G_1(s)[R_1(s) - H_1(s)Y_1(s)] + G_2(s)G_3(s)E_2(s) \ . \end{aligned}$$

Therefore,

$$Y_1(s) = \frac{G_1(s)G_2(s)}{1 + G_1(s)G_2(s)H_1(s)}R_1(s) + \frac{G_2(s)G_3(s)}{1 + G_1(s)G_2(s)H_1(s)}E_2(s) \ .$$

And, computing $E_2(s)$ (with $R_2(s) = 0$) we find

$$E_2(s) = H_2(s)Y_2(s) = H_2(s)G_6(s)\left[\frac{G_4(s)}{G_2(s)}Y_1(s) + G_5(s)E_2(s)\right]$$

or

$$E_2(s) = \frac{G_4(s)G_6(s)H_2(s)}{G_2(s)(1 - G_5(s)G_6(s)H_2(s))}Y_1(s) \ .$$

Substituting $E_2(s)$ into equation for $Y_1(s)$ yields

$$\begin{aligned} Y_1(s) &= \frac{G_1(s)G_2(s)}{1 + G_1(s)G_2(s)H_1(s)}R_1(s) \\ &+ \frac{G_3(s)G_4(s)G_6(s)H_2(s)}{(1 + G_1(s)G_2(s)H_1(s))(1 - G_5(s)G_6(s)H_2(s))}Y_1(s) \ . \end{aligned}$$

Finally, solving for $Y_1(s)$ yields

$$Y_1(s) = T_1(s)R_1(s)$$

where

$$T_1(s) = \left[\frac{G_1(s)G_2(s)(1 - G_5(s)G_6(s)H_2(s))}{(1 + G_1(s)G_2(s)H_1(s))(1 - G_5(s)G_6(s)H_2(s)) - G_3(s)G_4(s)G_6(s)H_2(s)}\right] \ .$$

Similarly, for $Y_2(s)$ we obtain

$$Y_2(s) = T_2(s)R_1(s) \ .$$

where

$$T_2(s) = \left[\frac{G_1(s)G_4(s)G_6(s)}{(1 + G_1(s)G_2(s)H_1(s))(1 - G_5(s)G_6(s)H_2(s)) - G_3(s)G_4(s)G_6(s)H_2(s)}\right] \ .$$

P2.33 The signal flow graph shows three loops:

$$L_1 = -G_1G_3G_4H_2$$
$$L_2 = -G_2G_5G_6H_1$$
$$L_3 = -H_1G_8G_6G_2G_7G_4H_2G_1 \ .$$

The transfer function Y_2/R_1 is found to be

$$\frac{Y_2(s)}{R_1(s)} = \frac{G_1G_8G_6\Delta_1 - G_2G_5G_6\Delta_2}{1-(L_1+L_2+L_3)+(L_1L_2)} \ ,$$

where for path 1

$$\Delta_1 = 1$$

and for path 2

$$\Delta_2 = 1 - L_1 \ .$$

Since we want Y_2 to be independent of R_1, we need $Y_2/R_1 = 0$. Therefore, we require

$$G_1G_8G_6 - G_2G_5G_6(1+G_1G_3G_4H_2) = 0 \ .$$

P2.34 The closed-loop transfer function is

$$\frac{Y(s)}{R(s)} = \frac{G_3(s)G_1(s)(G_2(s)+K_5K_6)}{1-G_3(s)(H_1(s)+K_6)+G_3(s)G_1(s)(G_2(s)+K_5K_6)(H_2(s)+K_4)} \ .$$

P2.35 The equations of motion are

$$m_1\ddot{y}_1 + b(\dot{y}_1-\dot{y}_2) + k_1(y_1-y_2) = 0$$
$$m_2\ddot{y}_2 + b(\dot{y}_2-\dot{y}_1) + k_1(y_2-y_1) + k_2y_2 = k_2x$$

Taking the Laplace transform yields

$$(m_1s^2+bs+k_1)Y_1(s) - (bs+k_1)Y_2(s) = 0$$
$$(m_2s^2+bs+k_1+k_2)Y_2(s) - (bs+k_1)Y_1(s) = k_2X(s)$$

Therefore, after solving for $Y_1(s)/X(s)$, we have

$$\frac{Y_2(s)}{X(s)} = \frac{k_2(bs+k_1)}{(m_1s^2+bs+k_1)(m_2s^2+bs+k_1+k_2)-(bs+k_1)^2} \ .$$

P2.36 (a) We can redraw the block diagram as shown in Figure P2.36. Then,

$$T(s) = \frac{K_1/s(s+1)}{1+K_1(1+K_2s)/s(s+1)} = \frac{K_1}{s^2+(1+K_2K_1)s+K_2} \ .$$

CHAPTER 3

State Variable Models

Exercises

E3.1 One possible set of state variables is

(a) the current i_{L_2} through L_2,

(b) the voltage v_{C_2} across C_2, and

(c) the current i_{L_1} through L_1.

We can also choose v_{C_1}, the voltage across C_1 as the third state variable, in place of the current through L_1.

E3.2 We know that the velocity is the derivative of the position, therefore we have

$$\frac{dy}{dt} = v \ ,$$

and from the problem statement

$$\frac{dv}{dt} = -k_1 v(t) - k_2 y(t) + k_3 i(t) \ .$$

This can be written in matrix form as

$$\frac{d}{dt}\begin{pmatrix} y \\ v \end{pmatrix} = \begin{bmatrix} 0 & 1 \\ -k_2 & -k_1 \end{bmatrix}\begin{pmatrix} y \\ v \end{pmatrix} + \begin{bmatrix} 0 \\ k_3 \end{bmatrix} i \ .$$

Define $u = i$, and let $k_1 = k_2 = 1$. Then,

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}u$$

where

$$\mathbf{A} = \begin{bmatrix} 0 & 1 \\ -1 & -1 \end{bmatrix} \ , \quad \mathbf{B} = \begin{bmatrix} 0 \\ k_3 \end{bmatrix} \ , \text{and} \quad \mathbf{x} = \begin{pmatrix} y \\ v \end{pmatrix} \ .$$

E3.3 The charactersitic roots, denoted by λ, are the solutions of $\det(\lambda \mathbf{I} - \mathbf{A}) = 0$. For this problem we have

$$\det(\lambda \mathbf{I} - \mathbf{A}) = \det \left(\begin{bmatrix} \lambda & -1 \\ 1 & (\lambda+1) \end{bmatrix} \right) = \lambda(\lambda+1) + 1 = \lambda^2 + \lambda + 1 = 0 \ .$$

Therefore, the characteristic roots are

$$\lambda_1 = -\frac{1}{2} + j\frac{\sqrt{3}}{2} \quad \text{and} \quad \lambda_2 = -\frac{1}{2} - j\frac{\sqrt{3}}{2} \ .$$

E3.4 The system in phase variable form is

$$\begin{aligned} \dot{\mathbf{x}} &= \mathbf{A}\mathbf{x} + \mathbf{B}u \\ y &= \mathbf{C}\mathbf{x} \end{aligned}$$

where

$$\mathbf{A} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -8 & -6 & -4 \end{bmatrix} , \quad \mathbf{B} = \begin{bmatrix} 0 \\ 0 \\ 20 \end{bmatrix} , \quad \mathbf{C} = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} \ .$$

E3.5 From the block diagram we determine that the state equations are

$$\begin{aligned} \dot{x_2} &= -(fk+d)x_1 + ax_1 + fu \\ \dot{x_1} &= -kx_2 + u \end{aligned}$$

and the output equation is

$$y = bx_2 \ .$$

Therefore,

$$\begin{aligned} \dot{\mathbf{x}} &= \mathbf{A}\mathbf{x} + \mathbf{B}u \\ y &= \mathbf{C}\mathbf{x} + \mathbf{D}u \ , \end{aligned}$$

where

$$\mathbf{A} = \begin{bmatrix} 0 & -k \\ a & -(fk+d) \end{bmatrix} , \quad \mathbf{B} = \begin{bmatrix} 1 \\ f \end{bmatrix} , \quad \mathbf{C} = \begin{bmatrix} 0 & b \end{bmatrix} \text{ and } \quad \mathbf{D} = [0] \ .$$

E3.6 (a) The state transition matrix is

$$\mathbf{\Phi}(t) = e^{\mathbf{A}t} = \mathbf{I} + \mathbf{A}t + \frac{1}{2!}\mathbf{A}^2 t^2 + \cdots$$

In state variable form we have

$$\dot{\mathbf{x}} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -\frac{(k+k_1)}{m} & -\frac{b}{m} & \frac{k_1}{m} & 0 \\ 0 & 0 & 0 & 1 \\ \frac{k_1}{m} & 0 & -\frac{(k+k_1)}{m} & -\frac{b}{m} \end{bmatrix} \mathbf{x}$$

where $x_1 = x, x_2 = \dot{x}, x_3 = q$ and $x_4 = \dot{q}$.

E3.16 The governing equations of motion are

$$m_1\ddot{x} + k_1(x-q) + b_1(\dot{x}-\dot{q}) = u(t)$$
$$m_2\ddot{q} + k_2 q + b_2\dot{q} + b_1(\dot{q}-\dot{x}) + k_1(q-x) = 0 \ .$$

Let $x_1 = x, x_2 = \dot{x}, x_3 = q$ and $x_4 = \dot{q}$. Then,

$$\dot{\mathbf{x}} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -\frac{k_1}{m_1} & -\frac{b_1}{m_1} & \frac{k_1}{m_1} & \frac{b_1}{m_1} \\ 0 & 0 & 0 & 1 \\ \frac{k_1}{m_2} & \frac{b_1}{m_2} & -\frac{(k_1+k_2)}{m_2} & -\frac{(b_1+b_2)}{m_2} \end{bmatrix} \mathbf{x} + \begin{bmatrix} 0 \\ \frac{1}{m_1} \\ 0 \\ 0 \end{bmatrix} u(t) \ .$$

Since the output is $y(t) = q(t)$, then

$$y = \begin{bmatrix} 0 & 0 & 1 & 0 \end{bmatrix} \mathbf{x} \ .$$

E3.17 At node 1 we have

$$C_1\dot{v}_1 = \frac{v_a - v_1}{R_1} + \frac{v_2 - v_1}{R_2}$$

and at node 2 we have

$$C_2\dot{v}_2 = \frac{v_b - v_2}{R_3} + \frac{v_1 - v_2}{R_2} \ .$$

Let

$$x_1 = v_1$$

and

$$x_2 = v_2 \ .$$

Then, in matrix form we have

$$\dot{\mathbf{x}} = \begin{bmatrix} -\left(\frac{1}{R_1C_1} + \frac{1}{R_2C_1}\right) & \frac{1}{R_2C_1} \\ -\frac{1}{R_2C_2} & -\left(\frac{1}{R_3C_2} + \frac{1}{R_2C_2}\right) \end{bmatrix} \mathbf{x} + \begin{bmatrix} \frac{1}{R_1C_1} & 0 \\ 0 & \frac{1}{R_3C_2} \end{bmatrix} \begin{bmatrix} v_a \\ v_b \end{bmatrix} .$$

E3.18 The governing equations of motion are

$$Ri_1 + L_1\frac{di_1}{dt} + v = v_a$$
$$L_2\frac{di_2}{dt} + v = v_b$$
$$i_L = i_1 + i_2 = C\frac{dv}{dt} .$$

Let $x_1 = i_1, x_2 = i_2, x_3 = v, u_1 = v_a$ and $u_2 = v_b$. Then,

$$\dot{\mathbf{x}} = \begin{bmatrix} -\frac{R}{L_1} & 0 & -\frac{1}{L_1} \\ 0 & 0 & -\frac{1}{L_2} \\ \frac{1}{C} & \frac{1}{C} & 0 \end{bmatrix} \mathbf{x} + \begin{bmatrix} \frac{1}{L_1} & 0 \\ 0 & \frac{1}{L_2} \\ 0 & 0 \end{bmatrix} \mathbf{u}$$

$$y = \begin{bmatrix} 0 & 0 & 1 \end{bmatrix} \mathbf{x} + [0]\,\mathbf{u} .$$

E3.19 First, compute the matrix

$$sI - \mathbf{A} = \begin{bmatrix} s & -1 \\ 3 & s+4 \end{bmatrix} .$$

Then, $\Phi(s)$ is

$$\Phi(s) = (s\mathbf{I} - \mathbf{A})^{-1} = \frac{1}{\Delta(s)} \begin{bmatrix} s+4 & 1 \\ -3 & s \end{bmatrix}$$

where $\Delta(s) = s^2 + 4s + 3$, and

$$G(s) = \begin{bmatrix} 10 & 0 \end{bmatrix} \begin{bmatrix} \frac{s+4}{\Delta(s)} & \frac{1}{\Delta(s)} \\ -\frac{3}{\Delta(s)} & \frac{s}{\Delta(s)} \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \frac{10}{s^2 + 4s + 3} .$$

E3.20 The linearized equation can be derived from the observation that $\sin\theta \approx \theta$ when $\theta \approx 0$. In this case, the linearized equations are

$$\ddot{\theta} + \frac{g}{L}\theta + \frac{k}{m}\dot{\theta} = 0 .$$

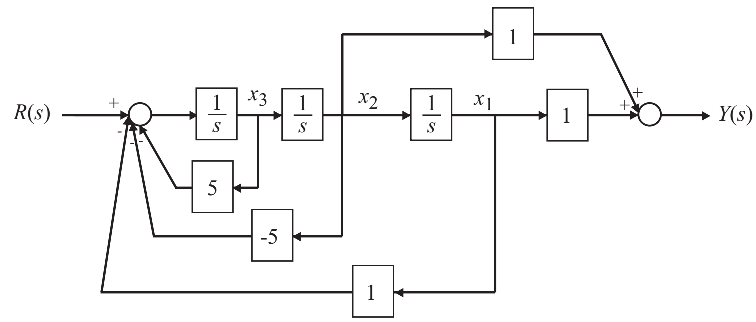

FIGURE P3.5
Block diagram model.

P3.6 The node equations are

$$
\begin{aligned}
0.00025\frac{dv_1}{dt} + i_L - \frac{v_i - v_1}{4000} &= 0 \\
0.0005\frac{dv_2}{dt} - i_L + \frac{v_2}{1000} - i_3 &= 0 \\
0.002\frac{di_L}{dt} + v_2 - v_1 &= 0 \ .
\end{aligned}
$$

Define the state variables

$$
x_1 = v_1 \quad x_2 = v_2 \quad x_3 = i_L \ .
$$

Then,

$$
\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}u
$$

where

$$
\mathbf{A} = \begin{bmatrix} -1 & 0 & -4000 \\ 0 & -2 & 2000 \\ 500 & -500 & 0 \end{bmatrix} , \quad \mathbf{B} = \begin{bmatrix} 1 & 0 \\ 0 & 2000 \\ 0 & 0 \end{bmatrix}
$$

P3.7 Given $K = 1$, we have

$$
KG(s) \cdot \frac{1}{s} = \frac{(s+1)^2}{s(s^2+1)} \ .
$$

We then compute the closed-loop transfer function as

$$
T(s) = \frac{s^2 + 2s + 1}{3s^3 + 5s^2 + 5s + 1} = \frac{s^{-1} + 2s^{-2} + s^{-3}}{3 + 5s^{-1} + 5s^{-2} + s^{-3}} \ .
$$

(c) The characteristic equation is

$$\det[s\mathbf{I} - \mathbf{A}] = \det \begin{bmatrix} s & -1 & 0 \\ 0 & s & -1 \\ 5 & 1 & s+10.1 \end{bmatrix} = s^3 + 10.1s^2 + s + 5 = 0 \ .$$

The roots of the characteristic equation are

$$s_1 = -10.05 \quad \text{and} \quad s_{2,3} = -0.0250 \pm 0.7049j \ .$$

All roots lie in the left hand-plane, therefore, the system is stable.

P3.10 (a) From the signal flow diagram, we determine that a state-space model is given by

$$\dot{\mathbf{x}} = \begin{bmatrix} -K_1 & K_2 \\ -K_1 & -K_2 \end{bmatrix} \mathbf{x} + \begin{bmatrix} K_1 & -K_2 \\ K_1 & K_2 \end{bmatrix} \begin{bmatrix} r_1 \\ r_2 \end{bmatrix}$$

$$\mathbf{y} = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \mathbf{x} \ .$$

(b) The characteristic equation is

$$\det[s\mathbf{I} - \mathbf{A}] = s^2 + (K_2 + K_1)s + 2K_1K_2 = 0 \ .$$

(c) When $K_1 = K_2 = 1$, then

$$\mathbf{A} = \begin{bmatrix} -1 & 1 \\ -1 & -1 \end{bmatrix} .$$

The state transition matrix associated with $\mathbf{A}$ is

$$\mathbf{\Phi} = \mathcal{L}^{-1}\left\{[s\mathbf{I} - \mathbf{A}]^{-1}\right\} = e^{-t} \begin{bmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{bmatrix} .$$

P3.11 The state transition matrix is

$$\Phi(t) = \begin{bmatrix} (2t-1)e^{-t} & -2te^{-t} \\ 2te^{-t} & (-2t+1)e^{-t} \end{bmatrix} .$$

So, when $x_1(0) = x_2(0) = 10$, we have

$$\mathbf{x}(t) = \Phi(t)x(0)$$

or

$$x_1(t) = 10e^{-t}$$
$$x_2(t) = 10e^{-t}$$

P3.12 (a) The phase variable representation is

$$\dot{\mathbf{x}} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -48 & -44 & -12 \end{bmatrix} \mathbf{x} + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} r$$

$$y = [40 \quad 8 \quad 0]\mathbf{x} \ .$$

(b) The canonical representation is

$$\dot{\mathbf{z}} = \begin{bmatrix} -2 & 0 & 0 \\ 0 & -4 & 0 \\ 0 & 0 & -6 \end{bmatrix} \mathbf{z} + \begin{bmatrix} -0.5728 \\ 4.1307 \\ 4.5638 \end{bmatrix} r$$

$$y = [-5.2372 \quad -0.4842 \quad -0.2191]\mathbf{z}$$

(c) The state transition matrix is

$$\Phi(t) = \left[\ \Phi_1(t) \vdots \Phi_2(t) \vdots \Phi_3(t) \ \right] ,$$

where

$$\Phi_1(t) = \begin{bmatrix} e^{-6t} - 3e^{-4t} + 3e^{-2t} \\ -6e^{-6t} + 12e^{-4t} - 6e^{-2t} \\ 36e^{-6t} - 48e^{-4t} + 12e^{-2t} \end{bmatrix} \quad \Phi_2(t) = \begin{bmatrix} \frac{3}{4}e^{-6t} - 2e^{-4t} + \frac{5}{4}e^{-2t} \\ -\frac{9}{2}e^{-6t} + 8e^{-4t} - \frac{5}{2}e^{-2t} \\ 27e^{-6t} - 32e^{-4t} + 5e^{-2t} \end{bmatrix}$$

$$\Phi_3(t) = \begin{bmatrix} \frac{1}{8}e^{-6t} - \frac{1}{4}e^{-4t} + \frac{1}{8}e^{-2t} \\ -\frac{3}{4}e^{-6t} + e^{-4t} - \frac{1}{4}e^{-2t} \\ \frac{9}{2}e^{-6t} - 4e^{-4t} + \frac{1}{2}e^{-2t} \end{bmatrix} .$$